The General Circulation of the Atmosphere.

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§ 1. *Introduction.*

The difference in temperature between the equator and the poles of the earth tends to produce a movement of the atmosphere from the poles towards the equator at the earth's surface, and thus set up a steady circulation. The problem has been investigated by Ferrel* and Oberbeck.† The former did not attempt to make a complete analytical solution. The latter assumed that the atmosphere was incompressible but that it satisfied Boyle's law for a perfect gas, and he surrounded the atmosphere with a spherical boundary of undetermined height. In the present paper the known hydrodynamical equations for a viscous gas are first determined, using polar coordinates, by considering the flow of matter and momentum through a small volume. By finding the flow of energy another equation is obtained, which is used instead of the ordinary adiabatic assumption. This equation is equivalent to the equation of energy in the kinetic theory of gases but is here obtained independently of that theory. Making use of the fact that the height of the atmosphere is small compared with the earth's radius, an approximation to the motion is found for a non-rotating earth. The effect of the earth's rotation on the east and west motion is next considered, and then the resulting modification of the pressure gradient. Finally the solution which is obtained is compared with the observed facts.

§ 2. *Coordinates and Variables.*

In discussing the general circulation of the atmosphere on a spherical earth, it is convenient to use polar coordinates; namely: r, the distance of any point of the atmosphere from the earth's center; θ, the colatitude; and ϕ, the longitude. The variables which we wish to determine as functions of r, θ, ϕ, and t, the time, are: p, ρ, τ, the pressure, density, and absolute temperature; and u, v, w, the components of the velocity.

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* Ferrel: Recent Advances in Meteorology.

† Oberbeck: Papers on the Mechanics of the Earth's Atmosphere, translated by Cleveland Abbé.

The atmosphere is supposed to be a perfect gas, so that

$$p = R\rho\tau, \tag{1}$$

where R is a constant. Five other equations are necessary to determine the six variables p, ρ, τ, u, v, w. They are derived in the next three sections by considering the flow of matter, of momentum, and of energy.

§ 3. *The Flow of Matter.*

Consider the flow of matter through the six faces of a small volume $dr\,.\,rd\theta\,.\,r\sin\theta\,d\phi$ bounded by the spheres r, $r + dr$, the cones θ, $\theta + d\theta$, and the planes ϕ, $\phi + d\phi$. The rate of increase of the matter in this volume is

$$\frac{\partial\rho}{\partial t}(dr\,.\,rd\theta\,.\,r\sin\theta\,d\phi) = -\frac{\partial}{\partial r}(\rho u\,.\,rd\theta\,.\,r\sin\theta\,d\phi)\,dr$$
$$-\frac{\partial}{\partial\theta}(\rho v\,.\,r\sin\theta\,d\phi\,.\,dr)\,d\theta - \frac{\partial}{\partial\phi}(\rho w\,.\,dr\,.\,rd\theta)\,d\phi.$$

Dividing by $dr\,.\,rd\theta\,.\,r\sin\theta\,d\phi$, we have the continuity equation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{r^2\partial r}(r^2\rho u) + \frac{\partial}{r\sin\theta\,\partial\theta}(\rho v\sin\theta) + \frac{\partial}{r\sin\theta\,\partial\phi}(\rho w) = 0. \tag{2}$$

§ 4. *The Flow of Momentum.*

Consider the flow of momentum through the same small volume. Momentum being a vector quantity it is necessary to take account of the changes in the direction of the components of the velocity as we consider the different faces of the polar element of volume. Thus u turns through the angle $d\theta$ on the plane ϕ and through $\sin\theta\,d\phi$ on the cone θ, while v and $v + dv$ meet at the angle $\cos\theta\,d\phi$ on the earth's axis.

The normal stresses on the faces P, Q, R, the tangential stresses S, T, U, and the external force F also generate momentum; moreover the stresses change their directions as we consider different faces. The rate of increase of momentum in the direction of r is therefore

$$\frac{\partial}{\partial t}(\rho u)\,dr\,.\,rd\theta\,.\,r\sin\theta\,d\phi =$$
$$-\frac{\partial}{\partial r}\{(\rho u\,.\,u + P)\,r\sin\theta\,d\phi\,.\,rd\theta\}\,dr - \frac{\partial}{\partial\theta}\{(\rho v\,.\,u + U)\,r\sin\theta\,d\phi\,.\,dr\}\,d\theta$$
$$-\frac{\partial}{\partial\phi}\{(\rho w\,.\,u + T)\,r\,d\theta\,dr\}\,d\phi + (\rho v.\,v + Q)\,r\sin\theta\,d\phi\,dr\,.\,d\theta$$
$$+ (\rho w\,.\,w + R)\,r\,d\theta\,dr\sin\theta\,d\phi + \rho\,F_r\,dr\,.\,r\,d\theta\,.\,r\sin\theta\,d\phi.$$

Hence we have the equation of motion:

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{r^2\partial r}\{(\rho u\,.\,u+P)r^2\}+\frac{\partial}{r\sin\theta\,\partial\theta}\{(\rho v\,.\,u+U)\sin\theta\}$$
$$+\frac{\partial}{r\sin\theta\,\partial\phi}(\rho w\,.\,u+T)-\frac{\rho v\,.\,v+Q+\rho w\,.\,w+R}{r}-\rho F_r=0. \qquad (3)$$

Similarly,

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{r^2\partial r}\{(\rho u\,.\,v+U)r^2\}+\frac{\partial}{r\sin\theta\,\partial\theta}\{(\rho v\,.\,v+Q)\sin\theta\}$$
$$+\frac{\partial}{r\sin\theta\,\partial\phi}(\rho w\,.\,v+S)-\frac{\rho v\,.\,u+U-(\rho w\,.\,w+R)\cot\theta}{r}-\rho F_\theta=0 \qquad (4)$$

and

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{r^2\partial r}\{(\rho u\,.\,w+T)r^2\}+\frac{\partial}{r\sin\theta\,\partial\theta}\{(\rho v\,.\,w+S)\sin\theta\}$$
$$+\frac{\partial}{r\sin\theta\,\partial\phi}(\rho w\,.\,w+R)+\frac{\rho w\,.\,u+T+(\rho w\,.\,v+S)\cot\theta}{r}-\rho F_\phi=0. \qquad (5)$$

If $\frac{\partial\rho}{\partial t}$ is eliminated by means of (2), these equations can easily be identified with the ordinary form of the equations of motion in polar coordinates.

In Stokes' theory of viscosity the stresses P, Q, R, S, T, U are linear functions of the strains e, f, g, a, b, c; namely:

$$\left.\begin{aligned} P&=p+\tfrac{2}{3}\mu\delta-2\mu e,\\ Q&=p+\tfrac{2}{3}\mu\delta-2\mu f,\\ R&=p+\tfrac{2}{3}\mu\delta-2\mu g,\\ S&=-\mu a,\\ T&=-\mu b,\\ U&=-\mu c;\end{aligned}\right\} \qquad (6)$$

where the dilatation $\delta=e+f+g$; the three extensions are:

$$\left.\begin{aligned} e&=\frac{\partial u}{\partial r},\\ f&=\frac{\partial v}{r\partial\theta}+\frac{u}{r},\\ g&=\frac{\partial w}{r\sin\theta\,\partial\phi}+\frac{u}{r};\end{aligned}\right\} \qquad (7)$$

and the three shears:

$$\left.\begin{aligned} a &= \frac{\partial v}{r \sin\theta\, \partial\phi} + \frac{\partial w}{r\,\partial\theta} - \frac{w}{r}\cot\theta, \\ b &= \frac{\partial u}{r \sin\theta\, \partial\phi} + \frac{\partial w}{\partial r} - \frac{w}{r}, \\ c &= \frac{\partial u}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}. \end{aligned}\right\} \tag{8}$$

On substituting the values of the stresses in the equations of motion we could obtain them in the form which is generally used, but it is more convenient for our purpose to leave them in their present form.

§ 5. *The Flow of Energy.*

By considering the flow of matter and momentum we have obtained the equations of continuity and motion. In an exactly similar way the kinetic energy per unit of volume is $\frac{1}{2}\rho(u^2 + v^2 + w^2)$, and like ρ it is a scalar quantity. The energy in the form of heat is $C\rho\tau$, where C is the capacity of the atmosphere for heat. The rate of conduction of heat is $-\varkappa \operatorname{grad} \tau$, where $\varkappa$ is the conductivity. The stresses P, Q, R, S, T, U, and the external force F also do work and thus contribute energy. Taking account of all these different forms of energy, we find for the rate of increase of energy in a small polar element of volume:

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2) + C\rho\tau\right\} dr\,.\,r\,d\theta\,.\,r\sin\theta\,d\phi = \\ &-\frac{\partial}{\partial r}\left[\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2)u + C\rho\tau u + Pu + Uv + Tw - \varkappa\frac{\partial\tau}{\partial r}\right\} r\,d\theta\, r\sin\theta\,d\phi\right] dr \\ &-\frac{\partial}{\partial\theta}\left[\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2)v + C\rho\tau v + Uu + Qv + Sw - \varkappa\frac{\partial\tau}{r\,\partial\theta}\right\} r\sin\theta\,d\phi\,dr\right] d\theta \\ &-\frac{\partial}{\partial\phi}\left[\left\{\tfrac{1}{2}\rho(u^2+v^2+w^2)w + C\rho\tau w + Tu + Sv + Rw - \varkappa\frac{\partial\tau}{r\sin\theta\,\partial\phi}\right\} r\,d\theta\,dr\right] d\phi \\ &+\rho(F_r u + F_\theta v + F_\phi w)\,dr\,r\,d\theta\,r\sin\theta\,d\phi. \end{aligned}$$

Using (2) to eliminate $\frac{\partial\rho}{\partial t}$, substituting for the three components of F from (3), (4) and (5), and using (7) and (8), we find

$$C\rho\frac{d\tau}{dt} + Pe + Qf + Rg + Sa + Tb + Uc$$
$$-\frac{\partial}{r^2\partial r}\left(\varkappa r^2\frac{\partial\tau}{\partial r}\right) - \frac{\partial}{r\sin\theta\,\partial\theta}\left(\varkappa\frac{\partial\tau}{r\,\partial\theta}\sin\theta\right) - \frac{\partial}{r\sin\theta\,\partial\phi}\left(\varkappa\frac{\partial\tau}{r\sin\theta\,d\phi}\right) = 0.$$

On substituting the values of the stresses from (6) this equation becomes

$$C\rho\frac{d\tau}{dt} + p\delta + \frac{2}{3}\mu\delta^2 - 2\mu(e^2+f^2+g^2) - \mu(a^2+b^2+c^2)$$
$$-\frac{\partial}{r^2\partial r}\left(\varkappa r^2\frac{\partial\tau}{\partial r}\right) - \frac{\partial}{r\sin\theta\,\partial\theta}\left(\varkappa\frac{\partial\tau}{r\,\partial\theta}\sin\theta\right) - \frac{\partial}{r\sin\theta\,\partial\phi}\left(\varkappa\frac{\partial\tau}{r\sin\theta\,\partial\phi}\right) = 0.$$

This result agrees with the equation of energy in the kinetic theory of gases, but is here derived independently.

Replacing p and δ by the use of (1), (2) and (7), we obtain the more convenient form

$$C\rho\frac{d\tau}{dt} - R\tau\frac{d\rho}{d\tau} + \frac{2}{3}\mu\delta^2 - 2\mu(e^2+f^2+g^2) - \mu(a^2+b^2+c^2)$$
$$-\frac{\partial}{r^2\partial r}\left(\varkappa r^2\frac{\partial\tau}{\partial r}\right) - \frac{\partial}{r\sin\theta\,\partial\theta}\left(\varkappa\frac{\partial\tau}{r\,\partial\theta}\sin\theta\right) - \frac{\partial}{r\sin\theta\,\partial\phi}\left(\varkappa\frac{\partial\tau}{r\sin\theta\,\partial\phi}\right) = 0. \quad (9)$$

The conductivity and viscosity of the atmosphere are both small; hence, if the motion is so slow that we may neglect squares of velocities, (9) becomes

$$C\rho\frac{d\tau}{dt} - R\tau\frac{d\rho}{d\tau} = 0.$$

Integrating, we find that

$$\tau^C \propto \rho^R,$$

which gives, from (1),

$$p \propto \rho^{\frac{R+C}{C}},$$

the adiabatic law.

Since R is about $.4\,C$, we have

$$p = m\rho^{1.4}, \quad (10)$$

where m is a constant.

§ 6. *Adiabatic Equilibrium.*

When the temperature is a function of r only, the equations of motion become

$$\frac{\partial p}{\partial r} + g\rho = 0,$$

or, from (10),

$$1.4\, m\rho^{\cdot 4} \frac{d\rho}{dr} + g\rho = 0.$$

Integrating, we have

$$\frac{1.4}{.4} m\rho^{\cdot 4} + gr = \text{constant};$$

that is,

$$\frac{1.4}{.4} R\tau + gr = \text{constant}.$$

At the surface of the earth r is a and τ is τ_0; therefore

$$\frac{1.4}{.4} R(\tau - \tau_0) + g(r - a) = 0.$$

Hence τ is 0 when

$$r = a + \frac{1.4}{.4g} R\tau_0. \tag{11}$$

Substituting numerical values, we find that the height of the atmosphere is $\frac{1}{220}a$, which will be denoted by ah, where $h = \frac{1}{220}$.

If we now put $r = a(1 + h - x)$, so that ax is the distance from the top of the atmosphere, then τ is a linear function of x which vanishes when $x = 0$ and has the value τ_0 when $x = h$.

Hence, we have

$$\tau = \tau_0 \left(\frac{x}{h}\right); \tag{12}$$

and therefore, from (1) and (10),

$$p = p_0 \left(\frac{x}{h}\right)^{\frac{7}{2}} \tag{13}$$

and

$$\rho = \rho_0 \left(\frac{x}{h}\right)^{\frac{5}{2}}. \tag{14}$$

A relation between the constants is furnished by (1) and (11); namely,

$$g = \frac{7 p_0}{2ah\rho_0}. \tag{15}$$

§ 7. *Circulation on a Non-Rotating Earth.*

Assume that the temperature of the atmosphere decreases symmetrically from the equator to the poles and diminishes uniformly from the surface upwards. Then, instead of (11), we have

$$\tau = \frac{x}{h}(\tau_0 - \tau_2 \cos^2\theta), \tag{16}$$

where $\frac{\tau_2}{\tau_0}$ is small.

To satisfy (1), and from the form of the equations of motion, we therefore take instead of (13) and (14):

$$\left.\begin{aligned} p &= p_0\left(\frac{x}{h}\right)^{\frac{7}{2}} + p_2\cos^2\theta, \\ \rho &= \rho_0\left(\frac{x}{h}\right)^{\frac{5}{2}} + \rho_2\cos^2\theta, \end{aligned}\right\} \tag{17}$$

where

$$\frac{p_2}{p_0\left(\frac{x}{h}\right)^{\frac{7}{2}}} = \frac{\rho_2}{\rho_0\left(\frac{x}{h}\right)^{\frac{5}{2}}} - \frac{\tau_2}{\tau_0}. \tag{18}$$

Equilibrium is no longer possible, because, from (6) and (17),

$$\frac{\partial Q}{\partial \theta} \neq 0$$

and (4) is therefore not satisfied.

§ 8. *Solution of the Continuity Equation.*

When the motion is steady and symmetrical, the continuity equation takes the form

$$\frac{\partial}{r^2\partial r}(\rho r^2 u) + \frac{\partial}{r\sin\theta\,\partial\theta}(\rho v\sin\theta) = 0,$$

or approximately, from (14),

$$\left(\frac{x}{h}\right)^{\frac{5}{2}}\left\{-\frac{\partial u}{\partial x} + \frac{\partial}{\sin\theta\,\partial\theta}(v\sin\theta)\right\} - \frac{5}{2h}\left(\frac{x}{h}\right)^{\frac{3}{2}}u = 0.$$

Hence

$$\frac{\partial u}{\partial x} + \frac{5}{2}\frac{u}{x} = \frac{\partial}{\sin\theta\,\partial\theta}(v\sin\theta).$$

The solution of this equation, which corresponds to (17), is

$$\left.\begin{aligned} u &= U(1 - 3\cos^2\theta), \\ v &= V \sin\theta \cos\theta, \end{aligned}\right\} \tag{19}$$

where U and V are functions of x which satisfy

$$\frac{dU}{dx} + \frac{5}{2}\frac{U}{x} = -V. \tag{20}$$

§ 9. *Approximate Form of the Equations of Motion.*

Since x is a small ratio which lies between 0 and h (that is, between 0 and $\frac{1}{220}$), we see from (20) that U is small compared with V. Therefore, if we retain in the equations of motion only the most important terms, we find

$$\frac{\partial p}{\partial r} + g\rho = 0, \tag{21}$$

$$\frac{\partial p}{r\,\partial\theta} - \frac{\partial}{r^2\,\partial r}\left(\mu \frac{\partial v}{\partial r} r^2\right) = 0. \tag{22}$$

From (15), (17) and (19) these equations become

$$\frac{\partial p_2}{\partial x} = \frac{7}{2}\frac{p_0\,\rho_2}{h\,\rho_0} \tag{23}$$

and

$$2\,a\,p_2 = -\frac{\partial}{\partial x}\left(\mu \frac{\partial V}{\partial x}\right).$$

Since the viscosity varies as the .76 power of the absolute temperature, the latter equation may be written in the form

$$2\,a\,p_2 = -\frac{\partial}{\partial x}\left(\mu_0 \left(\frac{x}{h}\right)^{\frac{3}{4}} \frac{\partial V}{\partial x}\right). \tag{24}$$

§ 10. *Solution of the Equations of Motion.*

Finding first p from (18) and (21), next V from (24), and then U from (20), we have

$$U = Ax^{-\frac{5}{2}} + Bx^{\frac{23}{4}} + Cx^{\frac{23}{4}} \log x + Dx^{\frac{5}{4}} + Ex, \tag{25}$$

where A, B, D and E are arbitrary constants, but C must be so chosen that (18) is satisfied.

§ 11. *The Boundary Conditions.*

At the top of the atmosphere (where x is zero) it is assumed that u is zero and that the tangential stress $\mu \frac{\partial v}{\partial r}$ is zero; hence A and D are zero.

At the surface of the earth u and v are both zero; hence

$$U = B\left(\frac{19}{4} x^{\frac{23}{4}} \log \frac{x}{h} - x^{\frac{23}{4}} + h^{\frac{19}{4}} x\right). \tag{26}$$

Substituting this value of U in (18), we find

$$u = \frac{896\, a\, p_0 \tau_2}{107217\, h^{\frac{11}{4}} \mu_0 \tau_0} \left(\frac{19}{4} x^{\frac{23}{4}} \log \frac{x}{h} - x^{\frac{23}{4}} + h^{\frac{19}{4}} x\right)(1 - 3\cos^2\theta), \tag{27}$$

$$v = -\frac{896\, a\, p_0 \tau_2}{107217\, p^{\frac{11}{4}} \mu_0 \tau_0} \left(\frac{627}{16} x^{\frac{19}{4}} \log \frac{x}{h} - \frac{7}{2} x^{\frac{19}{4}} + \frac{7}{2} h^{\frac{19}{4}}\right) \sin\theta \cos\theta, \tag{28}$$

$$p = p_0 \left(\frac{x}{h}\right)^{\frac{7}{2}} \left\{1 + \left(\frac{7}{2} \log \frac{x}{h} + \frac{119}{99}\right) \cos^2\theta\right\}, \tag{29}$$

$$\rho = \rho_0 \left(\frac{x}{h}\right)^{\frac{5}{2}} \left\{1 + \left(\frac{7}{2} \log \frac{x}{h} + \frac{218}{99}\right) \cos^2\theta\right\}. \tag{30}$$

§ 12. *East and West Motion on a Rotating Earth.*

When the rotation of the earth is taken into account, it is convenient to denote by w the east and west velocity relative to the earth so that the actual velocity is $w + \varepsilon r \sin\theta$, ε being the earth's angular velocity about its axis. Substituting this value in (5) and retaining only the most important terms, we have

$$0 = -2\varepsilon\rho u \sin\theta - 2\varepsilon\rho v \cos\theta + \frac{\partial}{\partial r}\left(\mu \frac{\partial w}{\partial r}\right);$$

hence, using (19),

$$\frac{\partial}{\partial x}\left(x^{\frac{3}{4}} \frac{\partial w}{\partial x}\right) = \frac{2a^2 \varepsilon \rho_0}{\mu_0 h^{\frac{7}{4}}} \{U \sin\theta (1 - 3\cos^2\theta) + V \sin\theta \cos^2\theta\}.$$

The term containing U is small, compared with the term containing V, except near the equator, where $\cos\theta$ is small. Substituting from (27) and (28) for U and V, integrating twice and remembering that at the top of the atmosphere

(where x is zero) the tangential stress $\mu \frac{\partial w}{\partial r}$ is zero and that at the surface of the earth (where $x = h$) w is zero, we find

$$
\left.
\begin{aligned}
w = \frac{1792\, a^8 \varepsilon p_0 \rho_0 \tau_2}{107217\, h^{\frac{9}{2}} \mu_0^2 \tau_0} \Bigg[\Bigg\{ & \frac{2}{37} x^{\frac{19}{2}} \log \frac{x}{h} - \frac{596}{19.37^2} x^{\frac{19}{2}} + \frac{8}{171} h^{\frac{19}{4}} x^{\frac{19}{4}} \\
& - \frac{5588}{171.37^2} h^{\frac{19}{2}} \Bigg\} \sin\theta\, (1 - 3\cos^2\theta) \\
- \Bigg\{ \frac{19}{34} x^{\frac{17}{2}} \log \frac{x}{h} - \frac{53}{289} x^{\frac{17}{2}} & + \frac{4}{15} h^{\frac{19}{4}} x^{\frac{15}{4}} - \frac{361}{4335} h^{\frac{17}{2}} \Bigg\} \sin\theta \cos^2\theta \Bigg].
\end{aligned}
\right\} \quad (32)
$$

§ 13. *Pressure Gradient Due to the East and West Motion.*

When the rotation of the earth is taken into account, the equations of motion (21) and (22) become

$$0 = -\frac{\partial p}{\partial r} - g\rho + 2\varepsilon\rho w \sin\theta, \quad (33)$$

$$0 = -\frac{\partial p}{r\,\partial\theta} + 2\varepsilon\rho w \cos\theta + \frac{\partial}{r^2\,\partial r}\left(\mu r^2 \frac{\partial v}{\partial r}\right). \quad (34)$$

We must therefore replace (17) by

$$
\left.
\begin{aligned}
\rho &= \rho_0 \left(\frac{x}{h}\right)^{\frac{5}{2}} + \rho_2 \cos^2\theta + \rho_4 \cos^4\theta, \\
p &= p_0 \left(\frac{x}{h}\right)^{\frac{7}{2}} + p_2 \cos^2\theta + p_4 \cos^4\theta,
\end{aligned}
\right\} \quad (35)
$$

in which, from the law of gases, as in (18),

$$\frac{p_4}{p_0 \left(\frac{x}{h}\right)^{\frac{7}{2}}} = \frac{\rho_4}{\rho_0 \left(\frac{x}{h}\right)^{\frac{5}{2}}}. \quad (36)$$

Again, (19) must be altered to

$$
\begin{aligned}
u &= U(1 - 3\cos^2\theta) + U_4(3\cos^2\theta - 5\cos^4\theta), \\
v &= V \sin\theta\cos\theta + V_4 \sin\theta\cos^3\theta,
\end{aligned}
$$

where (from the continuity equation)

$$\frac{\partial U_4}{\partial x} + \frac{5}{2}\frac{U_4}{x} = -V_4. \quad (37)$$

From (34) we see that p_4 is of the same order as ρw in x, while V_4 is of order two higher in x; and therefore, from (37), U_4 is of order three higher in x. Hence (33) becomes

$$0 = -\frac{\partial p}{\partial r} - g\rho.$$

Using (36) and (15) we therefore find, on integration,

$$p_4 = P x^{\frac{7}{2}},$$

where P is an arbitrary constant which will be subsequently determined so that the boundary conditions are satisfied.

Substituting in (34), we have

$$0 = 4Px^{\frac{7}{2}} \sin\theta \cos^3\theta + 2a\varepsilon\rho w \cos\theta + \frac{\mu_0}{ah^{\frac{3}{4}}} \frac{\partial}{\partial x}\left(x^{\frac{3}{4}} \frac{\partial v}{\partial x}\right) \sin\theta \cos^3\theta.$$

Substituting the value of w from (32) and retaining only the second term in that equation, we have

$$0 = 4Px^{\frac{7}{2}} - \frac{3584\,a^4\varepsilon^2 p_0 \rho_0^2 \tau_2}{107217\,h^7 \mu_0^2 \tau_0}\left\{\frac{19}{34}x^{11}\log\frac{x}{h} - \frac{53}{289}x^{11} + \frac{4}{15}h^{\frac{19}{4}}x^{\frac{25}{4}} - \frac{361}{4335}h^{\frac{17}{2}}x^{\frac{5}{2}}\right\}$$
$$+ \frac{\mu_0}{ah^{\frac{3}{4}}}\frac{\partial}{\partial x}\left(x^{\frac{3}{4}}\frac{\partial V_4}{\partial x}\right).$$

Integrating with the boundary condition that, when x is zero, the tangential stress $\mu\dfrac{\partial v}{\partial x}$ is zero, we find

$$0 = \frac{8}{9}Px^{\frac{9}{2}} - \frac{3584\,a^4\varepsilon^2 p_0\rho_0^2\tau_2}{107217\,h^7\mu_0^2\tau_0}\left\{\frac{19}{408}x^{12}\log\frac{x}{h} - \frac{1595}{288\,.\,289}x^{12} + \frac{16}{435}h^{\frac{19}{4}}x^{\frac{29}{4}}\right.$$
$$\left. - \frac{722}{30345}x^{\frac{7}{2}}h^{\frac{17}{2}}\right\} + \frac{\mu_0}{ah^{\frac{3}{4}}}x^{\frac{3}{4}}\frac{\partial V_4}{\partial x}.$$

Dividing by $x^{\frac{3}{4}}$ and integrating with the boundary condition that, when x is h, V_4 is zero, we have

$$0 = \frac{32}{171}P\left(x^{\frac{19}{4}} - h^{\frac{19}{4}}\right) - \frac{3584\,a^4\varepsilon^2 p_0\rho_0^2\tau_2}{107217\,h^7\mu_0^2\tau_0}\left\{\frac{19}{4998}x^{\frac{49}{4}}\log\frac{x}{h} - \frac{93559}{72\,.\,17^2\,.\,49^2}x^{\frac{49}{4}}\right.$$
$$\left. + \frac{32}{4525}h^{\frac{19}{4}}x^{\frac{15}{2}} - \frac{2888}{455175}h^{\frac{17}{2}}x^{\frac{15}{4}} - \frac{87,822,265}{52200\,.\,17^2\,.\,49^2}h^{\frac{49}{4}}\right\} + \frac{\mu_0}{ah^{\frac{3}{4}}}V_4.$$

Substituting the value of V_4 in (37), multiplying by $x^{\frac{5}{2}}$, integrating with the boundary condition that U_4 is zero when x is zero, and, finally, putting x equal to h, we have, as the condition that U_4 is zero, the equation to determine P:

$$0 = \frac{32}{171} P\left(\frac{4}{33} - \frac{2}{7}\right) h^{\frac{33}{4}} - \frac{3584\, a^4 \varepsilon^2 p_0 \rho_0^2 \tau_2}{107217\, h^7 \mu_0^2 \tau_0} \left\{ - \frac{19}{4998} \frac{4^2}{63^2} - \frac{93559}{72 . 17^2 . 49^2} \frac{4}{63} \right.$$
$$\left. + \frac{32}{6525} \frac{1}{11} - \frac{2888}{455175} \frac{4}{29} - \frac{87,822,265}{52200 . 17^2 . 49^2} \frac{2}{7} \right\} h^{\frac{63}{4}}.$$

Hence

$$P = \frac{140.7\, a^4 \varepsilon^2 p_0 \rho_0^2 \tau_2 h^{\frac{1}{2}}}{107217\, \mu_0^2 \tau_0},$$

and therefore

$$p = p_0 \left(\frac{x}{h}\right)^{\frac{7}{2}} \left\{ 1 + \frac{\tau_2}{\tau_0}\left(\frac{7}{2} \log \frac{x}{h} + \frac{119}{99}\right) \cos^2\theta + \frac{140.7\, a^4 \varepsilon^2 p_0 \rho_0^2 \tau_2 h^{\frac{1}{2}}}{107217\, \mu_0^2 \tau_0} \cos^4\theta \right\}. \quad (38)$$

§ 14. *Discussion of the Results Obtained.*

The vertical velocity u_2 has a maximum value

$$\frac{383\, a p_0 \tau_2 h^3}{107217\, \mu_0 \tau_0}$$

when x is about $.55\,h$.

The southerly velocity v_2 has a maximum value

$$\frac{1023\, a p_0 \tau_2 h^2}{107217\, \mu_0 \tau_0}$$

when x is about $.886\,h$; and it changes to northerly when x is $.73\,h$.

It is seen from (32) that the velocity is easterly except near the equator, where, on account of the smallness of $\cos\theta$, the second term is important and the velocity is westerly.

The pressure p_2 has its maximum positive value,

$$\frac{119}{99} \frac{p_0 \tau_2}{\tau_0},$$

when x is h. It vanishes when x is $.71\,h$ and has its greatest negative value when x is $.53\,h$.

§ 15. *The Observed Facts in the Atmosphere.*

It is known that* the temperature at the surface of the earth is

$$8.5 - 20.95 \cos^2\theta,$$

or, in our notation,

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* Ferrel: Recent Advances in Meteorology, p. 452.

$$\tau = 303 - 42\cos^2\theta,$$
$$= \tau_0 - \tau_2\cos^2\theta;$$

so that $\frac{\tau_2}{\tau_0}$ is not very small. It is also known that*

$$p = 758 + 31\cos^2\theta - 61\cos^4\theta.$$

Putting $x = h$ in (38), the value obtained was

$$p = p_0 + \frac{119\, p_0 \tau_2}{99\,\tau_0}\cos^2\theta - \frac{140.7\, a^4 \varepsilon^2 p_0 \rho_0^2 \tau_2 h^{\frac{1}{2}}}{107217\,\mu_0^2 \tau_0}\cos^4\theta;$$

hence our value for p_2 appears to be too large, although its value decreases very rapidly as we ascend vertically. As regards p_4, if we substitute numerical values we find too large a value. The explanation is probably that, on account of the large amount of eddying motion in the atmosphere, the ordinary experimental value for μ is smaller than its effective value in the atmosphere. Moreover the motion we are considering is unstable except for small values of $\frac{\tau_2}{\tau_0}$ and large values of μ. There appears then to be only a qualitative and not a quantitative confirmation of the solution obtained.

As regards the velocities, it is agreed† that the velocity at the equator is westerly, and in the middle latitude easterly, and increasing with the altitude; but there is no evidence of a northerly motion in the upper atmosphere as theory would lead us to expect.

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* Overbeck: *loc. cit.*

† Hildebrandson: Brit. Assoc. Report, 1903.